<u>Fourier Series for the Periodic Waveform *x(t)*</u>

$$x(t) = X_{DC} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

where

$$X_{DC} = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2\pi}\int_0^{2\pi} x(\omega t)\, d(\omega t)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(\omega t) \cos n\omega t\, d(\omega t)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x(\omega t) \sin n\omega t\, d(\omega t)$$

$$T = \frac{2\pi}{\omega}$$

<u>Alternate Form for the Fourier Series</u>

$$x(t) = X_{DC} + \sum_{n=1}^{\infty} c_n \cos(n\omega t - \psi_n)$$

where

$$c_n = \sqrt{a_n^2 + b_n^2}$$

$$\psi_n = \tan^{-1}\left[\frac{b_n}{a_n}\right] \qquad \text{(four-quadrant arctangent)}$$

Waveform Attributes for the Periodic Waveform *x(t)*

DC (average) value:

$$X_{DC} = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2\pi}\int_0^{2\pi} x(\omega t)d(\omega t)$$

RMS value:

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\, dt} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} x^2(\omega t)\, d(\omega t)}$$

or in terms of Fourier components

$$X_{rms} = \sqrt{X_{DC}^2 + \frac{a_1^2}{2} + \frac{b_1^2}{2} + \frac{a_2^2}{2} + \frac{b_2^2}{2} + \ldots}$$

or

$$X_{rms} = \sqrt{X_{DC}^2 + \frac{c_1^2}{2} + \frac{c_2^2}{2} + \ldots}$$

Average power conveyed by the *v(t)*-*i(t)* pair:

$$P_{AVG} = \frac{1}{T}\int_0^T v(t)\, i(t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} v(\omega t)\, i(\omega t)\, d(\omega t)$$

or in terms of the Fourier components

$$P_{AVG} = V_{DC}I_{DC} + \frac{a_{v,1}\, a_{i,1}}{2} + \frac{b_{v,1}\, b_{i,1}}{2} + \frac{a_{v,2}\, a_{i,2}}{2} + \frac{b_{v,2}\, b_{i,2}}{2} + \ldots$$

or

$$P_{AVG} = V_{DC}I_{DC} + \frac{c_{v,1}\, c_{i,1}}{2}\cos(\psi_{v,1} - \psi_{i,1}) + \frac{c_{v,2}\, c_{i,2}}{2}\cos(\psi_{v,2} - \psi_{i,2}) + \ldots$$

<u>Useful Trigonometric Identities</u>

$\sin(-x) = -\sin(x) = \sin(x \pm \pi)$ and $\cos(-x) = \cos(x) = -\cos(x \pm \pi)$

$\sin x = \cos(x - \frac{\pi}{2})$

$\cos x = \sin(x + \frac{\pi}{2})$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \mp B) = \cos A \cos B \pm \sin A \sin B$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\sin A \pm \sin B = 2\sin\left[\frac{1}{2}(A \pm B)\right] \cdot \cos\left[\frac{1}{2}(A \mp B)\right]$

$\cos A + \cos B = 2\cos\left[\frac{1}{2}(A+B)\right] \cdot \cos\left[\frac{1}{2}(A-B)\right]$

$\cos A - \cos B = -2\sin\left[\frac{1}{2}(A+B)\right] \cdot \sin\left[\frac{1}{2}(A-B)\right]$

$\sin 2x = 2\sin x \cos x$

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$

$\sin\frac{x}{2} = \pm\sqrt{\frac{1}{2}(1-\cos x)}$ $\pm$ depends on the quadrant of $\frac{x}{2}$

$\cos\frac{x}{2} = \pm\sqrt{\frac{1}{2}(1+\cos x)}$ $\pm$ depends on the quadrant of $\frac{x}{2}$

$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$ $\pm$ depends on the quadrant of $\frac{x}{2}$

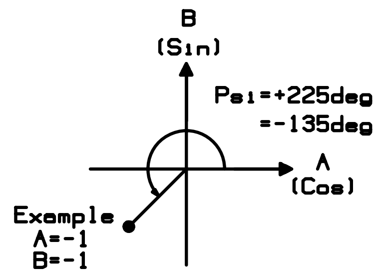
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$a\ \cos\omega t\ +\ b\ \sin\omega t\ =\ c\ \cos(\omega t - \psi)$$

where $c\ =\ \sqrt{a^2 + b^2}$

and $\psi\ =\ \tan^{-1}[\frac{b}{a}]$



Four-quadrant arc tangent (not principal value).

See example figure above; example below.

$$-\cos\omega t\ -\ \sin\omega t\ =\ \sqrt{2}\ \cos(\omega t\ -\ 225^\circ)\ =\ \sqrt{2}\ \cos(\omega t\ +\ 135^\circ)$$

$$\sum_{n=0}^{M-1}\ \cos(\omega t\ -\ n\tfrac{2\pi}{M})\ =\ \sum_{n=0}^{M-1}\ \sin(\omega t\ -\ n\tfrac{2\pi}{M})\ =\ 0$$ (complete phasor set identity, M phases)

$$\cos(\omega t)\ +\ \cos(\omega t - \tfrac{2\pi}{3})\ +\ \cos(\omega t - \tfrac{4\pi}{3})\ =\ 0$$ (the three-phase case)

Useful Integrals (n and m are integers)

$$\int \sin x\ \sin mx\ dx\ =\ \frac{\sin(n-m)x}{2(n-m)}\ -\ \frac{\sin(n+m)x}{2(n+m)}\ \text{ if } n^2 \neq m^2$$

$$\int \sin x\ \cos mx\ dx\ =\ \frac{-\cos(n-m)x}{2(n-m)}\ -\ \frac{\cos(n+m)x}{2(n+m)}\ \text{ if } n^2 \neq m^2$$

$$\int \cos nx\ \cos mx\ dx\ =\ \frac{\sin(n-m)x}{2(n-m)}\ +\ \frac{\sin(n+m)x}{2(n+m)}\ \text{ if } n^2 \neq m^2$$

$$\int x\ \sin(ax)\ dx\ =\ \frac{1}{a^2}\sin ax\ -\ \frac{x}{a}\cos ax$$

$$\int x\ \cos(ax)\ dx\ =\ \frac{1}{a^2}\cos ax\ +\ \frac{x}{a}\sin ax$$

$$\int \sin^2 nx\ dx\ =\ \frac{x}{2}\ -\ \frac{\sin 2nx}{4n}$$

$$\int \cos^2 nx\ dx\ =\ \frac{x}{2}\ +\ \frac{\sin 2nx}{4n}$$

$$\int \sin nx\ \cos nx\ dx\ =\ -\ \frac{\cos 2nx}{4n}$$

Useful Definite Integrals (*n* and *m* are integers)

$$\int_0^{2\pi} \sin nx \, \sin mx \, dx = \int_0^{2\pi} \sin nx \, \cos mx \, dx = \int_0^{2\pi} \cos nx \, \cos mx \, dx = 0 \text{ if } n^2 \neq m^2$$

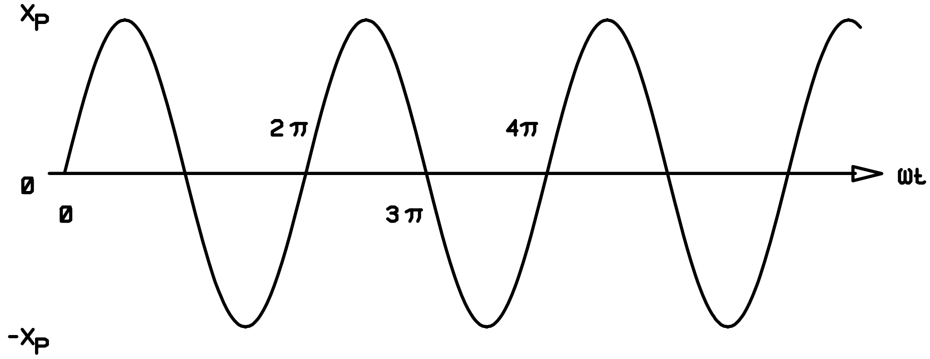
$$\int_0^{\pi} \sin^2 nx \, dx = \int_0^{\pi} \cos^2 nx \, dx = \tfrac{\pi}{2} \text{ for } n \neq 0$$

$$\int_0^{2\pi} \sin^2 nx \, dx = \int_0^{2\pi} \cos^2 nx \, dx = \pi \text{ for } n \neq 0$$

$$\int_0^{2\pi} \sin^2 nx \, dx = \int_0^{2\pi} \cos^2 nx \, dx = \pi \text{ for } n \neq 0$$

Attributes of Common Waveforms

A. Sine Wave $x(t)$

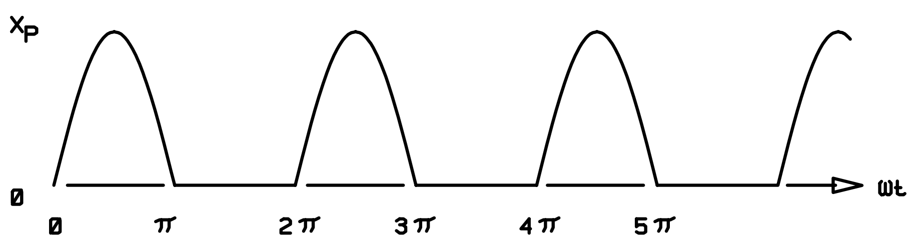


average value $X_{dc} = 0$

rms value $X_{rms} = \frac{X_P}{\sqrt{2}}$

Fourier series $x(t) = X_P \sin\omega t$

B. Half-Wave Rectified Sine Wave $x(t)$



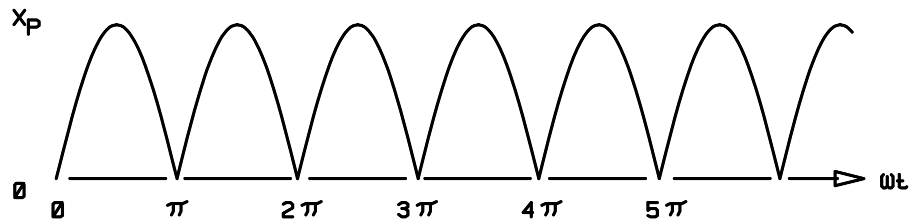
average value $X_{dc} = \frac{X_P}{\pi}$

rms value $X_{rms} = \frac{X_P}{2}$

Fourier series $x(t) = \frac{X_P}{\pi} + \frac{X_P}{2}\sin\omega t - \frac{2X_P}{\pi}\sum_{n=2}^{\infty}\frac{\cos n\omega t}{(n+1)(n-1)}$ n even only

$$= \frac{X_P}{\pi} + \frac{X_P}{2}\sin\omega t - \frac{2X_P}{\pi}\left[\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \frac{1}{35}\cos 6\omega t + \ldots\right]$$

C. Full-Wave Rectified Sine Wave $x(t)$



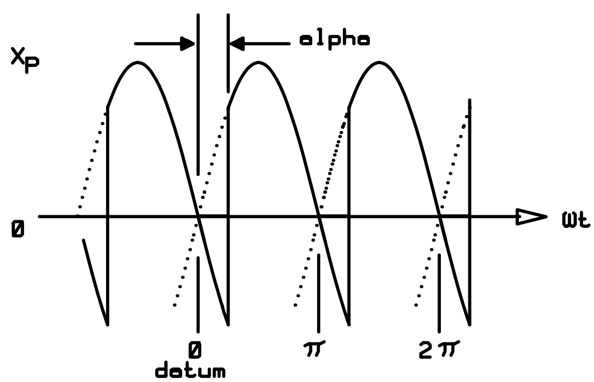
average value $X_{dc} = \frac{2X_P}{\pi}$

rms value $X_{rms} = \frac{X_P}{\sqrt{2}}$

Fourier series $x(t) = \frac{2X_P}{\pi} - \frac{4X_P}{\pi} \sum_{n=2}^{\infty} \frac{\cos n\omega t}{(n+1)(n-1)}$ n even only

$$= \frac{2X_P}{\pi} - \frac{4X_P}{\pi}\left[\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \frac{1}{35}\cos 6\omega t + \ldots\right]$$

D. Two-Pulse Controlled Rectifier Waveform $x(t)$



average value $X_{dc} = \frac{2X_P}{\pi}\cos\alpha$

rms value $X_{rms} = \frac{X_P}{\sqrt{2}}$

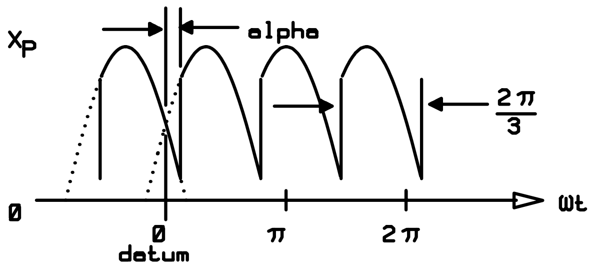
Fourier series

$$x(t) = \frac{2X_P}{\pi}\cos\alpha + \frac{2X_P}{\pi}\sum_{p=1}^{\infty}\cos p\pi\left\{\frac{\cos[2p\omega t + (2p+1)\alpha]}{2p+1} - \frac{\cos[2p\omega t + (2p-1)\alpha]}{2p-1}\right\}$$

or $x(t) = \frac{2X_P}{\pi}\cos\alpha + \frac{2X_P}{\pi}\sum_{p=1}^{\infty} c_{2p}\cos(2p\omega t + \phi_{2p})$

where $c_{2p} = \sqrt{\frac{1}{(2p+1)^2} - \frac{2\cos 2\alpha}{(2p+1)(2p-1)} + \frac{1}{(2p-1)^2}}$

E. Three-Pulse Controlled Rectifier Waveform $x(t)$



average value $X_{dc} = \frac{3\sqrt{3}\, X_P}{2\pi} \cos\alpha$

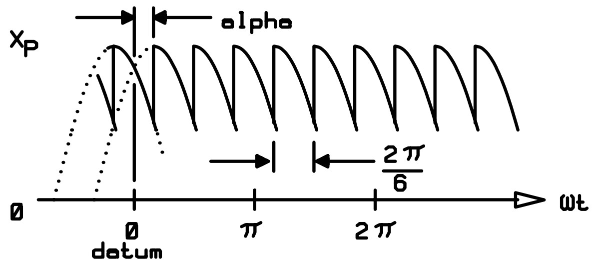
rms value $X_{rms} = \frac{3\sqrt{3}\, X_P}{2\pi} \sqrt{\frac{2\pi^2}{27} + \frac{\pi}{6\sqrt{3}} \cos 2\alpha}$

Fourier series $x(t) = \frac{3\sqrt{3}\, X_P}{2\pi} \cos\alpha + \frac{3\sqrt{3}\, X_P}{2\pi} \sum_{p=1}^{\infty} \cos p\pi \left\{ \frac{\cos[3p\omega t + (3p+1)\alpha]}{3p+1} - \frac{\cos[3p\omega t + (3p-1)\alpha]}{3p-1} \right\}$

or $x(t) = \frac{3\sqrt{3}\, X_P}{2\pi} \cos\alpha + \frac{3\sqrt{3}\, X_P}{2\pi} \sum_{p=1}^{\infty} c_{3p} \cos(3p\omega t + \phi_{3p})$

where $c_{3p} = \sqrt{\frac{1}{(3p+1)^2} - \frac{2\cos 2\alpha}{(3p+1)(3p-1)} + \frac{1}{(3p-1)^2}}$

F. Six-Pulse Controlled Rectifier Waveform $x(t)$



average value $X_{dc} = \frac{3X_P}{\pi}\cos\alpha$

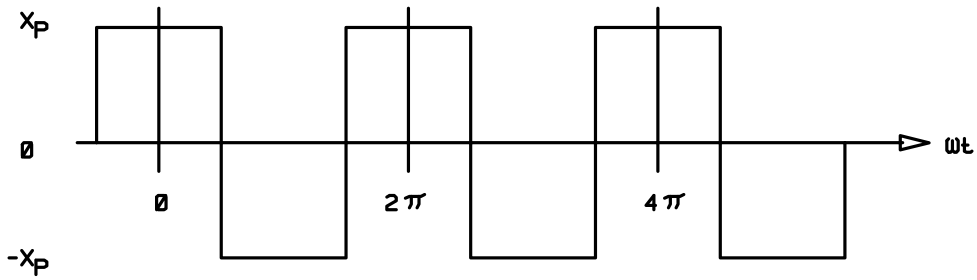
rms value $X_{rms} = \frac{3X_P}{\pi}\sqrt{\frac{\pi^2}{18} + \frac{\pi}{4\sqrt{3}}\cos 2\alpha}$

Fourier series $x(t) = \frac{3X_P}{\pi}\cos\alpha + \frac{3X_P}{\pi}\sum_{p=1}^{\infty}\left\{\frac{\cos[6p\omega t + (6p+1)\alpha]}{6p+1} - \frac{\cos[6p\omega t + (6p-1)\alpha]}{6p-1}\right\}$

or $x(t) = \frac{3X_P}{\pi}\cos\alpha + \frac{3X_P}{\pi}\sum_{p=1}^{\infty} c_{6p}\cos(6p\omega t + \phi_{6p})$

where $c_{6p} = \sqrt{\frac{1}{(6p+1)^2} - \frac{2\cos 2\alpha}{(6p+1)(6p-1)} + \frac{1}{(6p-1)^2}}$

G. Square Wave $x(t)$



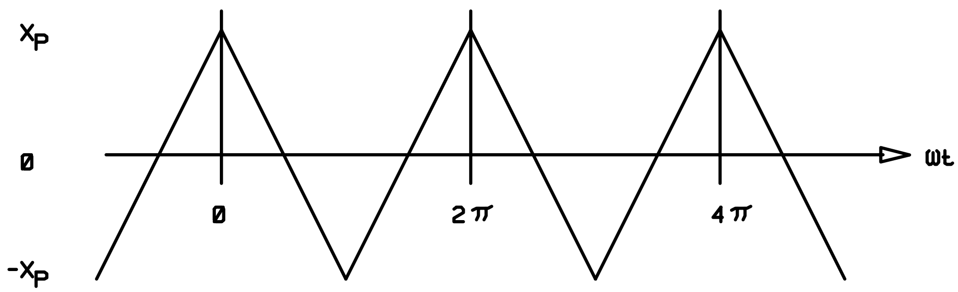
average value $X_{dc} = 0$

rms value $X_{rms} = X_P$

Fourier series $x(t) = 0 + \frac{4X_P}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left(n\frac{\pi}{2}\right)}{n} \cos n\omega t$ for n odd only

$$= \frac{4X_P}{\pi}\left[\cos \omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \frac{1}{7}\cos 7\omega t + \ldots\right]$$

H. Triangle Wave $x(t)$



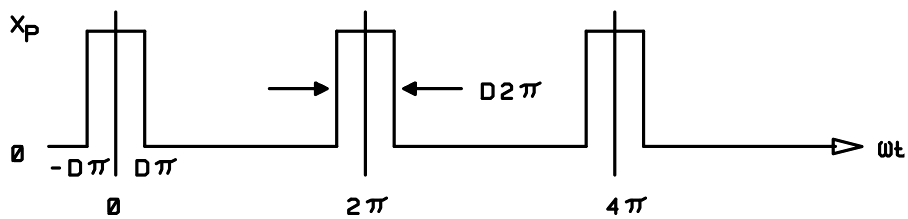
average value $X_{dc} = 0$

rms value $X_{rms} = \frac{X_P}{\sqrt{3}}$

Fourier series $x(t) = 0 + \frac{8X_P}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\left(n\frac{\pi}{2}\right)}{n^2} \cos n\omega t$ for n odd only

$$= \frac{8X_P}{\pi^2}\left[\cos \omega t - \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t - \frac{1}{49}\cos 7\omega t + \ldots\right]$$

I. Pulse Waveform $x(t)$ ("D" is the duty factor)



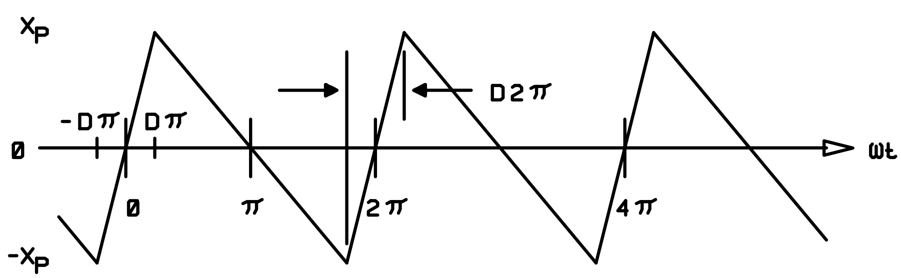
average value $X_{dc} = D\,X_P$

rms value $X_{rms} = \sqrt{D}\;X_P$

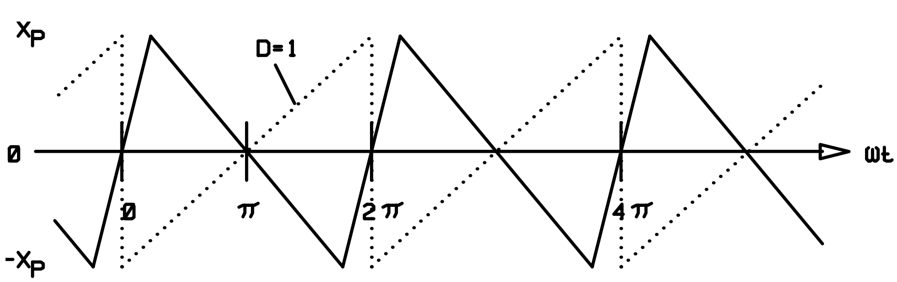
Fourier series $x(t) = DX_P + \frac{2X_P}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin nD\pi}{n}\right) \cos n\omega t$ for all n

$$= DX_P + \frac{2X_P}{\pi}\left[\sin D\pi \cdot \cos \omega t + \frac{\sin 2D\pi}{2} \cdot \cos 2\omega t + \frac{\sin 3D\pi}{3} \cdot \cos 3\omega t + \dots\right]$$

J. General Sawtooth Ramp Waveform $x(t)$ ("D" is the duty factor of the rising slope)



The "sweep" waveform is either of two special cases: D = 1, illustrated below, or D = 0, not shown.



average value $X_{dc} = 0$

rms value $X_{rms} = \frac{X_P}{\sqrt{3}}$

Fourier series $x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$ for all n

where

$$b_n = \frac{2X_P}{n^2\pi^2} \frac{\sin(nD\pi)}{D(1-D)}$$

and for the two special cases:

$$b_n = \frac{2X_P}{n\pi} \quad \text{for} \quad D = 0$$

$$b_n = -\frac{2X_P}{n\pi} \cos n\pi \quad \text{for} \quad D = 1$$

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